Mathematics and Statistics
(math114)

College of Business Administration
Jazan University

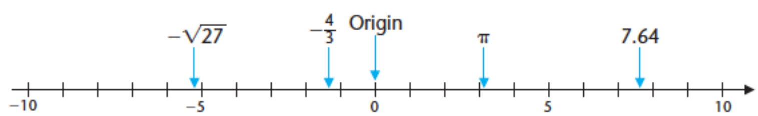
Chapter R: Basic Algebraic Operations

R-1 Algebra and Real Numbers

➢ ***The Set of Real Numbers:***

Symbol	Name	Description	Examples
N	Natural numbers	Counting numbers (also called positive integers)	1, 2, 3, . . .
Z	Integers	Natural numbers, their negatives, and 0 (also called whole numbers)	. . . , −2, −1, 0, 1, 2, . . .
Q	Rational numbers	Numbers that can be represented as a/b, where a and b are integers and $b \neq 0$; decimal representations are repeating or terminating	$-4, 0, 1, 25, -\frac{3}{5}, \frac{2}{3}, 3.67, -0.33\overline{3}$,* $5.27\overline{27}$
I	Irrational numbers	Numbers that can be represented as nonrepeating and nonterminating decimal numbers	$\sqrt{2}, \pi, \sqrt[3]{7}$, 1.414213 . . . ,† 2.71828182 . . .†
R	Real numbers	Rational numbers and irrational numbers	

➢ ***The Real Number Line:***



➢ ***Addition and Multiplication of Real Numbers:***

› **DEFINITION 1** **Addition and Multiplication of Rationals**

For rational numbers a/b and c/d, where a, b, c, and d are integers and $b \neq 0$, $d \neq 0$:

Addition: $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Multiplication: $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Ex. Perform the indicated operations:

A. $\frac{1}{3} + \frac{6}{5} = \frac{(1)(5)+(3)(6)}{(3)(5)} = \frac{5+18}{15} = \frac{23}{15}$

B. $\frac{1}{3} + \frac{1}{5} = \frac{5+3}{15} = \frac{8}{15}$ ⟶ جمع توحيد مقامات

C. $\frac{3}{4} + \frac{4}{3} = \frac{9+16}{12} = \frac{25}{12}$ ⟶ جمع توحيد مقامات

D. $\frac{8}{3}\cdot\frac{5}{4} = \frac{(8)(5)}{(3)(4)} = \frac{40}{12} = \frac{10}{3}$ *because* $\frac{40=4\cdot 10}{12=4.3} = \frac{10}{3}$

E. $\frac{2}{3}\cdot\frac{4}{7} = \frac{8}{21}$ → ضرب.. تبسيط. بسط / مقام.مقام

F. $\left(-\frac{3}{5}\right)\left(-\frac{5}{3}\right) = \frac{15}{15} = 1$ → ضرب عدد مع معكوسه الضربي يساوي 1

➢ *Further Operations and Properties*

- ***Basic Properties of The Set of Real Numbers:***

Addition Properties & Multiplication Properties: These operations are

- **Commutative (+):** $x + y = y + x$ *ex.* $\frac{3}{2} + \frac{5}{7} = \frac{5}{7} + \frac{3}{2}$

  Commutative (.): $xy = yx$ *ex.* $\frac{3}{2}\cdot\frac{5}{7} = \frac{5}{7}\cdot\frac{3}{2}$

- **Associative (+):** $(x + y) + z = x + (y + z)$ *ex.* $\frac{3}{2} + \left(\frac{5}{7} + \frac{9}{4}\right) = \left(\frac{3}{2} + \frac{5}{7}\right) + \frac{9}{4}$

  Associative (.): $(x.y).z = x.(y.z)$ *ex.* $\frac{3}{2}\cdot\left(\frac{5}{7}\cdot\frac{9}{4}\right) = \left(\frac{3}{2}\cdot\frac{5}{7}\right)\cdot\frac{9}{4}$

- $\mathbf{0}$ ***is an additive identity*** (+): $0 + x = x + 0 = x$ *ex.* $0 + \frac{5}{2} = \frac{5}{2} + 0 = \frac{5}{2}$

  $\mathbf{1}$ ***is a multiplictive identity*** (.): $1.x = x.1 = x$ *ex.* $1.\frac{5}{2} = \frac{5}{2}.1 = \frac{5}{2}$

- **Additive inverse (+):**

  for each x ***in R***, $-x$ ***its*** *additive inverse*

  ; ***that is*** $x + (-x) = (-x) + x = 0$

  ex. $\frac{5}{4}$ ***its additive inverse*** $-\frac{5}{4}$; ***that is*** $\frac{5}{4} + \left(-\frac{5}{4}\right) = \frac{5}{4} - \frac{5}{4} = 0$

  Multiplicative inverse (.):

for each x ***in R***, $x \neq 0, x^{-1} = \frac{1}{x}$ ***is its multiplicative inverse***

; ***that is*** $xx^{-1} = x^{-1}x = 1$

ex. $\frac{5}{4}$ ***its a multiplicative inverse*** $(\frac{5}{4})^{-1} = \frac{4}{5}$; ***that is*** $\frac{5}{4}\cdot\frac{4}{5} = \frac{5.4}{4.5} = \frac{20}{20} = 1$

ex. $(-\frac{17}{9})^{-1} = -\frac{9}{17}$

- **Distributive:** $x(y + z) = xy + xz$

  $(x + y)z = xz + yz$

Ex. Which real number property justifies the indicated statement?

A. $(7x)y = 7(xy)$ ……associative…… → التجميع

B. $a(b + c) = ab + ac$ ……Commutative…… → تبديل

C. $a(b + c) = (b + c)a$ ……Distributive…… → توزيع

D. $(2x+3y)+5y=2x+(3y+5y)$ ...associative... تجميع
E. $(x+y)(a+b)=(x+y)a+(x+y)b$ ...Distributive... توزيع
F. *If* $a+b=0$ *, then* $b=-a$ ...inverse... المعكوس

THEOREM 3 Fraction Properties

For all real numbers a, b, c, d, and k (division by 0 excluded):

1. $\frac{a}{b}=\frac{c}{d}$ if and only if $ad=bc$

   $\frac{4}{6}=\frac{6}{9}$ since $4\cdot 9=6\cdot 6$

2. $\frac{ka}{kb}=\frac{a}{b}$

   $\frac{7\cdot 3}{7\cdot 5}=\frac{3}{5}$

3. $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

   $\frac{3}{5}\cdot\frac{7}{8}=\frac{3\cdot 7}{5\cdot 8}=\frac{21}{40}$

4. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$

   $\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\cdot\frac{7}{5}=\frac{14}{15}$

5. $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$

   $\frac{3}{6}+\frac{4}{6}=\frac{3+4}{6}=\frac{7}{6}$

6. $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$

   $\frac{7}{8}-\frac{2}{8}=\frac{7-2}{8}=\frac{5}{8}$

7. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

   $\frac{2}{3}+\frac{1}{5}=\frac{2\cdot 5+3\cdot 1}{3\cdot 5}=\frac{13}{15}$

Ex. Perform the indicated operations:

A. $100\div 0=$ *Division by* 0 *never allowed*

B. $\frac{1}{2}+\frac{1}{7}=\frac{7+2}{14}=\frac{9}{14}$

C. $\frac{8}{9}-\frac{4}{5}=\frac{40-36}{45}=\frac{4}{45}$

D. $\left(-\frac{1}{10}\right).\frac{3}{8}=\frac{-3}{80}$

E. $0\div 0=$ كمية غير معروفه

F. $\frac{4}{7}\div\left(3-\frac{6}{2}\right)=\frac{4}{7}\div 0=$ كمية غير معروفه

G. $-(4^{-1}+3)=-\left(\frac{1}{4}+3\right)=-\left(\frac{1+12}{4}\right)=-\frac{13}{4}$

H. $\left(\frac{3}{8}\right)^{-1}+2^{-1}=\frac{8}{3}+\frac{1}{2}=\frac{(8)(2)+(3)(1)}{(3)(2)}=\frac{16+3}{6}=\frac{19}{6}$

I. $\left(-6+\frac{9}{2}\right)^{-1}=\left(\frac{-6}{1}+\frac{9}{2}\right)^{-1}=\left(\frac{(-6)(2)+(1)(9)}{(1)(2)}\right)^{-1}=\left(\frac{-12+9}{2}\right)^{-1}=\left(\frac{-3}{2}\right)^{-1}=\frac{2}{-3}$

R-2 Exponents and Radicals

- ***Integer Exponents:***

> **DEFINITION 1** a^n, n an Integer and a a Real Number
>
> For n a positive integer and a a real number:
>
> $a^n = a \cdot a \cdot \ldots \cdot a$ n factors of a
>
> $a^{-n} = \frac{1}{a^n}$ $(a \neq 0)$
>
> $a^0 = 1$ $(a \neq 0)$

Ex.

A. $2^5 = 2.2.2.2.2 = 32$

B. $7^{-3} = \frac{1}{7.^3} = \frac{1}{7.7.7} = \frac{1}{343}$

C. $23^0 = 1$

Ex. Write using positive exponents or decimals:

A. $(u^3v^2)^0 = 1$

B. $x^{-8} = \frac{1}{x^8}$

C. $10^{-3} = \frac{1}{10^3} = \frac{1}{(10)(10)(10)} = \frac{1}{1000} = 0.001$

D. $\frac{x^{-3}}{y^{-5}} = \frac{x^{-3}}{1} \cdot \frac{1}{y^{-5}} = \frac{1}{x^3} \cdot \frac{y^5}{1} = \frac{1(y^5)}{(x^3)(1)} = \frac{y^5}{x^3}$

> **THEOREM 1** Properties of Integer Exponents
>
> For n and m integers and a and b real numbers:
>
> 1. $a^m a^n = a^{m+n}$ $a^5a^{-7} = a^{5+(-7)} = a^{-2}$
> 2. $(a^n)^m = a^{mn}$ $(a^3)^{-2} = a^{(-2)3} = a^{-6}$
> 3. $(ab)^m = a^m b^m$ $(ab)^3 = a^3b^3$
> 4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $b \neq 0$ $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$
> 5. $\frac{a^m}{a^n} = \begin{cases} a^{m-n} \\ \frac{1}{a^{n-m}} \end{cases}$ $a \neq 0$ $\frac{a^3}{a^{-2}} = a^{3-(-2)} = a^5$, $\frac{a^3}{a^{-2}} = \frac{1}{a^{-2-3}} = \frac{1}{a^{-5}}$

Ex. Simplify using exponents properties, and express answer using positive exponents only?

A. $(3a^5)(2a^{-3}) = (3.2)(a^5a^{-3}) = 6a^{5+(-3)} = 6a^{5-3} = 6a^2$

B. $(2a^{-3}b^2)^{-2} = 2^{(1)(-2)}a^{(-3)(-2)}b^{(2)(-2)} = 2^{-2}a^6b^{-4} = \frac{1}{2^2}\frac{a^6}{1}\frac{1}{b^4} = \frac{1}{2.2}\frac{a^6}{b^4} = \frac{1}{4}\frac{a^6}{b^4} = \frac{a^6}{4b^4}$

C. $\frac{6x^{-2}}{8x^{-5}} = \frac{6}{8} \cdot \frac{x^{-2}}{x^{-5}} = \frac{3.2}{4.2} \cdot x^{-2-(-5)} = \frac{3}{4} \cdot x^{-2+5} = \frac{3}{4}x^3 = \frac{3}{4} \cdot \frac{x^3}{1} = \frac{3x^3}{4}$

D. $-4y^3 - (-4y)^3 = -4y^3 - (-4)^3y^3$

$= -4y^3 - (-4)(-4)(-4)y^3$
$= -4y^3 - (+16)(-4)y^3$
$= -4y^3 - (-64)y^3$
$= -4y^3 + 64y^3$
$= +60y^3$

Ex. evaluate each expression. If the answer is not an integer, write it in fraction form:

A. $3^7 = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 2187$

B. $\left(\frac{1}{2}\right)^8 = \frac{1^8}{2^8} = \frac{1}{256}$

C. $6^{-3} = \frac{1}{6^3} = \frac{1}{216}$

D. $(-5)^4 = 625$

E. $(-7)^{-2} = \frac{1}{-7^2} = \frac{1}{49}$

F. $-10 = -10$

➢ ***Roots of Real Numbers:***

› DEFINITION 2 Definition of an *n*th Root

For a natural number n and a and b real numbers:

a is an ***n*th root** of b if $a^n = b$ — 3 is a fourth root of 81, since $3^4 = 81$.

<u>Theorem</u>: Number of Real nth Roots of a Real Number b

Let n be a natural number and b a real number:

1. $\boldsymbol{b > 0}$: $If\ n\ is\ even, then\ b\ has\ two\ real\ nth\ roots, each\ the\ negative\ of\ the\ other;$
   Ex. $\sqrt{9} = \pm 3$
   $If\ n\ is\ odd, then\ b\ has\ one\ real\ nth\ root.$ ***Ex.*** $\sqrt[3]{8} = 2$

2. $If\ \boldsymbol{b = 0}$: $0\ is\ the\ only\ nth\ root\ of\ b = 0.$ ***Ex.*** $\sqrt{0} = 0$
   Ex. $\sqrt[3]{0} = 0$

3. $If\ \boldsymbol{b < 0}$: $If\ n\ is\ even, then\ b\ has\ no\ real\ nth\ root;$
   Ex. $\sqrt{-9} = \boldsymbol{undefined}$ (**not a real number**)
   $If\ n\ is\ odd, then\ b\ has\ one\ real\ nth\ root.$
   Ex. $\sqrt[3]{-8} = -2$

➢ ***Rational Exponents and Radicals:***

Ex. Evaluate each expression:

A. $b^{\frac{1}{n}} = \sqrt[n]{b}$

B. $9^{\frac{1}{2}} = \sqrt{9} = 3$

C. $\sqrt{121} = 121^{\frac{1}{2}} = 11$

D. $(-16)^{\frac{1}{4}} = \sqrt[4]{-16} = undefined\ (not\ a\ real\ number)$

E. $\sqrt[3]{-125} = (-125)^{\frac{1}{3}} = -5$

F. $(27)^{\frac{1}{3}} = \sqrt[3]{27} = 3$

G. $\sqrt[5]{32} = 32^{\frac{1}{5}} = 2$

Ex. Change to radical form:

A. $(100)^{\frac{1}{2}} = \sqrt{100} = 10$

B. $(32)^{\frac{1}{5}} = \sqrt[5]{32} = 2$

Ex. Change to rational exponent form:

A. $\sqrt{361} = (361)^{\frac{1}{2}} = 19$

B. $\sqrt[3]{x^2} + \sqrt[3]{y^2} = (x^2)^{\frac{1}{3}} + (y^2)^{\frac{1}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$

C. $4x\sqrt[5]{y^3} = 4x(y^3)^{\frac{1}{5}} = 4xy^{\frac{3}{5}}$

DEFINITION 4 $b^{m/n}$ **and** $b^{-m/n}$**, Rational Number Exponent**

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$b^{m/n} = (b^{1/n})^m \quad \text{and} \quad b^{-m/n} = \frac{1}{b^{m/n}}$$

$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$ $\quad$ $4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{8}$ $\quad$ $(-4)^{3/2}$ is not real

$(-32)^{3/5} = [(-32)^{1/5}]^3 = (-2)^3 = -8$

THEOREM 3 Rational Exponent/Radical Property

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$(b^{1/n})^m = (b^m)^{1/n} \quad \text{and} \quad (\sqrt[n]{b})^m = \sqrt[n]{b^m}$$

Ex. Simplify and express answers using positive exponents only?

A. $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 2.2 = 4$

B. $\sqrt[4]{3^{12}} = (3^{12})^{\frac{1}{4}} = 3^{\frac{12.1}{1.4}} = 3^{\frac{12}{4}} = 3^3 = 3.3.3 = 9.3 = 27$

C. $(3\sqrt[3]{x})(2\sqrt{x}) = \left(3x^{\frac{1}{3}}\right)\left(2x^{\frac{1}{2}}\right) = (3.2)x^{\frac{1}{3}+\frac{1}{2}} = 6x^{\frac{(1)(2)+(3)(1)}{(3)(2)}} = 6x^{\frac{2+3}{6}} = 6x^{\frac{5}{6}}$

D. $\left(\frac{4x^{\frac{1}{3}}}{x^{\frac{1}{2}}}\right)^{\frac{1}{2}} = \frac{4^{1\cdot\frac{1}{2}}x^{\frac{1}{3}\cdot\frac{1}{2}}}{x^{\frac{1}{2}\cdot\frac{1}{2}}} = \frac{\sqrt{4}x^{\frac{1}{6}}}{x^{\frac{1}{4}}} = 2x^{\frac{1}{6}-\frac{1}{4}} = 2x^{\frac{(1)(4)-(6)(1)}{(6)(4)}} = 2x^{\frac{4-6}{24}} = 2x^{\frac{-2}{24}} = 2x^{\frac{(-2)(1)}{(2)(12)}} = 2x^{\frac{-1}{12}} =$
$2.\frac{1}{x^{\frac{1}{12}}} = \frac{2}{x^{\frac{1}{12}}}$

➢ ***Simplifying Radicals:***

› THEOREM 4 Properties of Radicals

For n a natural number greater than 1, and x and y positive real numbers:

1. $\sqrt[n]{x^n} = x$ $\quad\sqrt[3]{x^3} = x$
2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\quad\sqrt[5]{xy} = \sqrt[5]{x}\sqrt[5]{y}$
3. $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ $\quad\sqrt[4]{\frac{x}{y}} = \frac{\sqrt[4]{x}}{\sqrt[4]{y}}$

Ex. Write in simplified radical form:

A. $\sqrt{16m^4y^8} = (16m^4y^8)^{\frac{1}{2}} = 16^{1\cdot\frac{1}{2}}m^{4\cdot\frac{1}{2}}y^{8\cdot\frac{1}{2}} = 16^{\frac{1}{2}}m^{\frac{4}{2}}y^{\frac{8}{2}} = \sqrt{16}m^2y^4 = 4m^2y^4$

B. $\sqrt{12x^5y^2} = (12x^5y^2)^{\frac{1}{2}}$
$= 12^{\frac{1}{2}}x^{\frac{5}{2}}y^{\frac{2}{2}}$
$= (3.4)^{\frac{1}{2}}\,x^2x^{\frac{1}{2}}\,y$
$= y3^{\frac{1}{2}}4^{\frac{1}{2}}\,x^2\,\sqrt{x}$
$= y\sqrt{3}\sqrt{4}\,x^2\,\sqrt{x}$
$= 2x^2y\sqrt{3x}$

C. $\sqrt[6]{16x^4y^2} = (16x^4y^2)^{\frac{1}{6}}$
$= 16^{\frac{1}{6}}x^{\frac{4}{6}}y^{\frac{2}{6}}$
$= (4^2)^{\frac{1}{6}}x^{\frac{2}{3}}y^{\frac{1}{3}}$
$= 4^{\frac{2}{6}}(x^2)^{\frac{1}{3}}\sqrt[3]{y}$
$= 4^{\frac{1}{3}}\,\sqrt[3]{x^2}\,\sqrt[3]{y}$
$= \sqrt[3]{4}\,\sqrt[3]{x^2}\,\sqrt[3]{y} = \sqrt[3]{4x^2y}$

D. $x\sqrt[5]{3^6x^7y^{11}} = x\,(3^6x^7y^{11})^{\frac{1}{5}}$
$= x\,(729)^{\frac{1}{5}}(x^7)^{\frac{1}{5}}(y^{11})^{\frac{1}{5}}$

$$= x\,(243.3)^{\frac{1}{5}}\, x^{\frac{7}{5}} y^{\frac{11}{5}}$$
$$= x\, x^{\frac{7}{5}}\, 243^{\frac{1}{5}}\, 3^{\frac{1}{5}}\, y^2 y^{\frac{1}{5}}$$
$$= x^{1+\frac{7}{5}}\, \sqrt[5]{243}\, \sqrt[5]{3}\, y^2\, \sqrt[5]{y}$$
$$= x^{\frac{5+7}{5}} .3 . y^2\, \sqrt[5]{3y}$$
$$= 3\, x^{\frac{12}{5}}\, y^2\, \sqrt[5]{3y}$$
$$= 3\, x^2\, x^{\frac{2}{5}}\, y^2\, \sqrt[5]{3y}$$
$$= 3\, x^2 y^2\, (x^2)^{\frac{1}{5}}\, \sqrt[5]{3y}$$
$$= 3\, x^2\, y^2\, \sqrt[5]{x^2}\, \sqrt[5]{3y}$$
$$= 3\, x^2\, y^2\, \sqrt[5]{3\, x^2\, y}$$

E. $\frac{6}{\sqrt{2x}} = \frac{6}{\sqrt{2x}} . \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{6\sqrt{2x}}{\left(\sqrt{2x}\right)^2} = \frac{6\sqrt{2x}}{2x} = \frac{3\sqrt{2x}}{x}$

F. $\frac{12y^2}{\sqrt{6y}} = \frac{12y^2}{\sqrt{6y}} . \frac{\sqrt{6y}}{\sqrt{6y}} = \frac{12y^2}{\left(\sqrt{6y}\right)^2} = \frac{12y^2}{6y} = \frac{(6.2)(y.y)}{6y} = 2y$

G. $\sqrt[3]{\frac{8x^4}{y}} = \left(\frac{8x^4}{y}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}(x^4)^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \frac{\sqrt[3]{8}x^{\frac{4}{3}}y^{\frac{2}{3}}}{y} = \frac{2x x^{\frac{1}{3}}(y^2)^{\frac{1}{3}}}{y} = \frac{2x\,\sqrt[3]{x}\,\sqrt[3]{y^2}}{y} = \frac{2x\,\sqrt[3]{xy^2}}{y}$

H. $-\sqrt{128} = -(128)^{\frac{1}{2}} = -(16.8)^{\frac{1}{2}} = -(16)^{\frac{1}{2}}(8)^{\frac{1}{2}} = -\sqrt{16}\,(4.2)^{\frac{1}{2}} = -4(4)^{\frac{1}{2}}(2)^{\frac{1}{2}} =$
$-4\sqrt{4}\,\sqrt{2} = -4.2.\sqrt{2} = -8\sqrt{2}$

I. $\sqrt{27} - 5\sqrt{3} = (27)^{\frac{1}{2}} - 5\sqrt{3} = (9.3)^{\frac{1}{2}} - 5\sqrt{3} = (9)^{\frac{1}{2}}(3)^{\frac{1}{2}} - 5\sqrt{3} = \sqrt{9}\sqrt{3} - 5\sqrt{3} =$
$3\sqrt{3} - 5\sqrt{3} = (3-5)\sqrt{3} = -2\sqrt{3}$

Ex. Simplify and express answer using positive exponents only:

A. $x^5 x^{-2} =$ $x^{5+-2} = x^3$

B. $(2y)(3y^2)(5y^4) =$ $2.3.5\, y^{1+2+4} = 30x^7$

C. $(a^2b^3)^5 =$ $a^{2.5} . b^{3.5} = a^{10} b^{15}$

D. $u^{\frac{1}{3}}\, u^{\frac{5}{3}} =$ $u^{\frac{1+5}{3}} = u^{\frac{6}{3}} = u^2$

E. $\left(\frac{w^4}{9x^{-2}}\right)^{-\frac{1}{2}} = \frac{w^{4(-\frac{1}{2})}}{9^{-\frac{1}{2}}\, x^{(-2)(-\frac{1}{2})}} = \frac{9^{\frac{1}{2}}\, w^{-\frac{4}{2}}}{x^{-\frac{2}{2}}} = \frac{\sqrt{9}\, w^{-2}}{x^{-1}} = \frac{3x}{w^2}$

R-3 Polynomials: Basic Operations and Factoring

➢ *Polynomials:*

Ex. Which of the following are polynomials?

A. $x^2 - 3x + 2$ ... $\boldsymbol{Polynomial}$ ...

B. $x^4 + \sqrt{2}$ ... $\boldsymbol{Polynomial}$ ...

C. $2x + 5 - \frac{1}{x}$ ... $\boldsymbol{NonPolynomial}$ ...

D. $\sqrt{x^3 - 4x + 1}$ ... $\boldsymbol{NonPolynomial}$ ...

Ex. Given the following polynomials what is the degree of the first term, second term, third term, and the whole polynomial:

A. $2x^3 - x^6 + 7$

The degree of the first term $= 3$
The degree of the second term $= 6$
The degree of the third term $= 0$
The degree of the whole polynomial $= 6$

B. $x^3y^2 + 2x^2y + 1$

The degree of the first term $= 5$
The degree of the second term $= 3$
The degree of the third term $= 0$
The degree of the whole polynomial $= 5$

Ex. Is the algebraic expression a polynomial? If so, give its degree.

A. $4 - x^2$ ...polynomil ... 2 → كثيرة حدود الدرجة 2
B. $x^5 - 4x^2 + 6^{-2}$ ...polynomil ... 5 → كثيرة حدود الدرجة 5
C. $x^4 + 3x - \sqrt{5}$ ...polynomil – 4 → كثيرة حدود الدرجة 4
D. $3x^4 - 2x^{-1} - 10$ ...No polynomil... ليست كثيرة حدود

➢ *Addition and Subtraction*

- ***Adding polynomials:***

Ex. Add: $x^4 - 3x^3 + x^2$, $-x^3 - 2x^2 + 3x$, ***and*** $3x^2 - 4x - 5$

Solution:

Vertically:

$$\begin{array}{r} x^4 - 3x^3 + x^2 \qquad\qquad \\ -x^3 - 2x^2 + 3x \qquad \\ 3x^2 - 4x - 5 \\ \hline x^4 - 4x^3 + 2x^2 - x - 5 \end{array}$$

Horizontally:

$(x^4 - 3x^3 + x^2) + (-x^3 - 2x^2 + 3x) + (3x^2 - 4x - 5)$
$= x^4 - 3x^3 + x^2 - x^3 - 2x^2 + 3x + 3x^2 - 4x - 5$
$= x^4 + (-3 - 1)x^3 + (1 - 2 + 3)x^2 + (3 - 4)x - 5$
$= x^4 - 4x^3 + 2x^2 - x - 5$

- ***Subtracting polynomials:***

Ex. Subtract: $4x^2 - 3x + 5 \quad from \quad x^2 - 8$

Solution:

Horizontally: $(x^2 - 8) - (4x^2 - 3x + 5) = x^2 - 8 - 4x^2 + 3x - 5$

$= (1 - 4)x^2 + 3x + (-8 - 5)$

$= -3x^2 + 3x - 13$

➢ *Multiplication*

- ***Multiplying polynomials:***

Ex. Multiply:

A. $(2x - 3)(3x^2 - 2x + 3)$

Solution: $= 2x(3x^2 - 2x + 3) - 3(3x^2 - 2x + 3)$

$= 2x(3x^2) + 2x(-2x) + 2x(3) - 3(3x^2) - 3(-2x) - 3(3)$

$= 6x^3 - 4x^2 + 6x - 9x^2 + 6x - 9$

$= 6x^3 + (-4 - 9)x^2 + (6 + 6)x - 9$

$= 6x^3 - 13x^2 + 12x - 9$

B. $(a - b)(a^2 + ab + b^2) =$ … … … … … … …

Ex. Perform the indicated operations and simplify

C. $(4x - y)^2 = (4x - y)(4x - y)$

$= 4x(4x - y) - y(4x - y)$

$= 4x(4x) + 4x(-y) - y(4x) - y(-y)$

$= 16x^2 - 4xy - 4xy + y^2$

$= 16x^2 - 8xy + y^2$

Or

$(4x - y)^2 = (4x)^2 - 2(4x)(y) + y^2$

$= 4^2x^2 - 8xy + y^2$

$= 16x^2 - 8xy + y^2$

D. $(5y - 1)(3 - 2y) =$ … … … … … … … … … ..

E. $2y - 3y[4 - 2(y - 1)] = 2y - 3y[4 - 2y + 2]$

$= 2y - 3y[6 - 2y]$

$= 2y - 3y(6) - 3y(-2y)$

$= 2y - 18y + 6y^2$

$= -16y + 6y^2$

F. $2(x - 1) + 3(2x - 3) - (4x - 5) =$ … … … … … … … ….

Ex. Remove Parenthesis and combine like terms:

A. $2(3x^2 - 2x + 5) + (x^2 + 3x - 7) = 2(3x^2) + 2(-2x) + 2(5) + x^2 + 3x - 7$

$= 6x^2 - 4x + 10 + x^2 + 3x - 7$

$= (6 + 1)x^2 + (-4 + 3)x + (10 - 7)$

$= 7x^2 - x + 3$

➢ *Factoring:*

Ex. $13 = 13.1 \quad (prime\ number)$

$30 = 6.5 = 2.3.5 \quad (composite\ number)$

$x^2 - 4 = (x - 2)(x + 2) \quad Not\ prime$

$x^2 - 2 = Prime\ because\ it\ can't\ be\ written\ as\ a\ product\ of\ two\ polynomials$

B) $(a-b)(a^2+ab+b^2)$ → هذا مفكوك فرق بين مكعبين

$a^3 - b^3$

بالضرب :.

$a(a^2+ab+b^2) - b(a^2+ab+b^2)$

$a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3$

$a^3 - b^3$

D) $(5y-1)(3-2y)$

$5y(3-2y) - 1(3-2y)$

$15y - 10y^2 - 3 + 2y$

$-10y^2 + 17y - 3$

F) $2(x-1) + 3(2x-3) - (4x-5)$

$(2x-2) + (6x-9) - (4x-5)$

$(8x-11) - (4x-5)$

$4x - 6$

- ***Factoring out common factors:***

Ex. Factor out, relative to the integers, all factors common to all terms:

A. $2x^3y - 8x^2y^2 - 6xy^3 = 2xy(x^2 - 4xy - 3y^2)$

B. $6x^4 - 8x^3 - 2x^2 =$ $2x^2(3x^2-4x-1)$

C. $x^2y + 2xy^2 + x^2y^2 =$ $xy(x+2y+xy)$

D. $2x(3x-2) - 7(3x-2) = (3x-2)(2x-7)$

E. $2w(y-2z) - x(y-2z) =$ $(y-2z)(2w-x)$

- ***Factoring by grouping:***

Ex. Factor completely, relative to the integers, by grouping:

A. $3x^2 - 6x + 4x - 8 = (3x^2 - 6x) + (4x - 8)$
$= 3x(x-2) + 4(x-2)$
$= (x-2)(3x+4)$

B. $wy + wz - 2xy - 2xz =$ $w(y+z)-2x(y+z) = (y+z)(w-2x)$

C. $x^2 + 4x + x + 4 = (x^2 + x) + (4x + 4)$
$= x(x+1) + 4(x+1)$
$= (x+1)(x+4)$

D. $3a^2 - 12ab - 2ab + 8b^2 = (3a^2 - 12ab) + (-2ab + 8b^2)$
$= 3a(a-4b) + 2b(-a+4b)$
$= 3a(a-4b) - 2b(a-4b)$
$= (a-4b)(3a-2b)$

E. $8ac + 3bd - 6bc - 4ad =$ $(4a-3b)(2c-d)$

Ex. Factor completely, relative to the integers:

A. $x^2 - xy + 3xy - 3y^2 = (x^2 - xy) + (3xy - 3y^2)$
$= x(x-y) + 3y(x-y)$
$= (x-y)(x+3y)$

- ***Factoring second (2[ed]) degree polynomials:***

Ex. Factor each polynomial, if possible, using integer coefficients:

A. $2x^2 + 3xy - 2y^2 = (2x-y)(x+2y)$

$\textbf{Check}$: $(2x-y)(x+2y) = 2x(x+2y) - y(x+2y)$
$= 2x^2 + 4xy - xy - 2y^2$
$= 2x^2 + 3xy - 2y^2$

B. $6x^2 + 5xy - 4y^2 = (3x+4y)(2x-1y)$

$\textbf{Check}$: $(3x+4y)(2x-y) = 3x(2x-y) + 4y(2x-y)$
$= 6x^2 - 3xy + 8xy - 4y^2$
$= 6x^2 + 5xy - 4y^2$

C. $x^2 - 3x + 4 = (x-2)(x-2)$ $\textbf{Check}$: $(x-2)(x-2) = x(x-2) - 2(x-2)$
$= x^2 - 2x - 2x + 4$

$$= x^2 - 4x + 4 \quad \times$$

$x^2 - 3x + 4 = (\boldsymbol{x} - \mathbf{4})(\boldsymbol{x} - \mathbf{1})$ ***Check***: $(\boldsymbol{x} - \mathbf{4})(\boldsymbol{x} - \mathbf{1}) = \boldsymbol{x}(\boldsymbol{x} - \mathbf{1}) - \mathbf{4}(\boldsymbol{x} - \mathbf{1})$

$$= \boldsymbol{x}^2 - \boldsymbol{x} - \mathbf{4}\boldsymbol{x} + \mathbf{4}$$

$$= x^2 - 5x + 4 \quad \times$$

$\therefore\ x^2 - 3x + 4$ *is not factorable.*

Ex. Factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

A. $2x^2 + x - 3 = (\mathbf{2}\boldsymbol{x} + \mathbf{3})(\boldsymbol{x} - \mathbf{1})$ ***Check***: $(\mathbf{2}\boldsymbol{x} + \mathbf{3})(\boldsymbol{x} - \mathbf{1}) = \mathbf{2}\boldsymbol{x}(\boldsymbol{x} - \mathbf{1}) + \mathbf{3}(\boldsymbol{x} - \mathbf{1})$

$$= \mathbf{2}\boldsymbol{x}^2 - \mathbf{2}\boldsymbol{x} + \mathbf{3}\boldsymbol{x} - \mathbf{3}$$

$$= 2x^2 + x - 3$$

B. $\boldsymbol{x}^2 + \mathbf{5}\boldsymbol{xy} - \mathbf{14}\boldsymbol{y}^2 =$ $(x+7y)(x-2y)$

C. $\mathbf{6}\boldsymbol{m}^2 - \boldsymbol{mn} - \mathbf{12}\boldsymbol{n}^2 =$ $(3m+4n)(2m-3n)$

- ***Factoring by using special factoring formulas:***

SPECIAL FACTORING FORMULAS

1. $u^2 + 2uv + v^2 = (u+v)^2$	**Perfect Square**
2. $u^2 - 2uv + v^2 = (u-v)^2$	**Perfect Square**
3. $u^2 - v^2 = (u-v)(u+v)$	**Difference of Squares**
4. $u^3 - v^3 = (u-v)(u^2 + uv + v^2)$	**Difference of Cubes**
5. $u^3 + v^3 = (u+v)(u^2 - uv + v^2)$	**Sum of Cubes**

Ex. Factor completely relative to the integers:

A. $\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{xy} + \mathbf{9}\boldsymbol{y}^2 = \boldsymbol{x}^2 + \mathbf{2}(\boldsymbol{x})(\mathbf{3}\boldsymbol{y}) + \mathbf{9}\boldsymbol{y}^2 = (\boldsymbol{x} + \mathbf{3}\boldsymbol{y})^2$

B. $\boldsymbol{x}^2 - \mathbf{6}\boldsymbol{xy} + \mathbf{9}\boldsymbol{y}^2 = \boldsymbol{x}^2 - \mathbf{2}(\boldsymbol{x})(\mathbf{3}\boldsymbol{y}) + \mathbf{9}\boldsymbol{y}^2 = (\boldsymbol{x} - \mathbf{3}\boldsymbol{y})^2$

C. $\mathbf{9}\boldsymbol{x}^2 - \mathbf{4}\boldsymbol{y}^2 = (\mathbf{3}\boldsymbol{x} - \mathbf{2}\boldsymbol{y})(\mathbf{3}\boldsymbol{x} - \mathbf{2}\boldsymbol{y})$

D. $\boldsymbol{x}^3 - \boldsymbol{y}^3 = (\boldsymbol{x} - \boldsymbol{y})(\boldsymbol{x}^2 + \boldsymbol{xy} + \boldsymbol{y}^2)$

E. $\boldsymbol{m}^3 + \boldsymbol{n}^3 = (\boldsymbol{m} + \boldsymbol{n})(\boldsymbol{m}^2 - \boldsymbol{mn} + \boldsymbol{n}^2)$

Ex. Factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

A. $\mathbf{16}\boldsymbol{x}^2 - \mathbf{25} =$ $(4x-5)(4x+5)$

B. $\mathbf{8}\boldsymbol{m}^3 - \mathbf{1} = \mathbf{2}^3\boldsymbol{m}^3 - \mathbf{1} = (\mathbf{2}\boldsymbol{m} - \mathbf{1})((\mathbf{2}\boldsymbol{m})^2 + \mathbf{2}\boldsymbol{m}(\mathbf{1}) + \mathbf{1}^2)$

$$= (\mathbf{2}\boldsymbol{m} - \mathbf{1})(\mathbf{2}^2\boldsymbol{m}^2 + \mathbf{2}\boldsymbol{m} + \mathbf{1}))$$

$$= (\mathbf{2}\boldsymbol{m} - \mathbf{1})(\mathbf{4}\boldsymbol{m}^2 + \mathbf{2}\boldsymbol{m} + \mathbf{1})$$

C. $\boldsymbol{x}^3 + \boldsymbol{y}^3\boldsymbol{z}^3 = (\boldsymbol{x} + \boldsymbol{y})(\boldsymbol{x}^2 - \boldsymbol{xyz} + \boldsymbol{y}^2\boldsymbol{z}^2)$

Chapter 1: Equations and Inequalities

1-1 Linear Equations and Applications

- ***Solving Linear Equations:***

DEFINITION 1 Linear Equation in One Variable

Any equation that can be written in the form

$$ax + b = 0 \qquad a \neq 0 \qquad \textbf{Standard Form}$$

where a and b are real constants and x is a variable, is called a **linear,** or **first-degree, equation** in one variable.

$5x - 1 = 2(x + 3)$ is a linear equation because after simplifying, it can be written in the standard form $3x - 7 = 0$.

Ex. Solve each equation:

A. $5x - 9 = 3x + 7$

$5x - 3x = 7 + 9$

$2x = 16$

$\frac{2x}{2} = \frac{16}{2}$

$x = 8$

Check: $5x - 9 = 3x + 7 \qquad (x = 8)$

$5(8) - 9 = 3(8) + 7$

$40 - 9 = 24 + 7$

$31 = 31$

Ex. Solve each equation:

A. $10x - 7 = 4x - 25$

$10x - 4x = -25 + 7$

$6x = -18$

$\frac{6x}{6x} = \frac{-18}{6}$

$x = -3$

Check: $10x - 7 = 4x - 25 \qquad (x = -3)$

$10(-3) - 7 = 4(-3) - 25$

$-30 - 7 = -12 - 25$

$-37 = -37$

B. $3(x + 2) = 5(x - 6)$

$3(x) + 3(2) = 5(x) + 5(-6)$

$3x + 6 = 5x - 30$

$3x - 5x = -30 - 6$

$-2x = -36$

$\frac{-2x}{-2} = \frac{-36}{-2}$

$x = +18$

Check: $3(x + 2) = 5(x - 6) \qquad (x = 18)$

$3(18 + 2) = 5(18 - 6)$

$3(20) = 5(12)$

$60 = 60$

C. $5 - \frac{3a-4}{5} = \frac{7-2a}{2}$

$(10)(5) - (10)\left(\frac{3a-4}{5}\right) = (10)\left(\frac{7-2a}{2}\right)$

$50 - 2(3a-4) = 5(7-2a)$

$50 - 6a + 8 = 35 - 10a$

$58 - 6a = 35 - 10a$

$10a - 6a = 35 - 58$

$4a = -23$

$a = \frac{-23}{4}$

Check: $5 - \frac{3a-4}{5} = \frac{7-2a}{2}$ $\quad\quad\quad\quad (a = \frac{-23}{4})$

$5 - \frac{1}{5}(3a-4) = \frac{1}{2}(7-2a)$

$5 - \frac{1}{5}\left(3\left(\frac{-23}{4}\right) - 4\right) = \frac{1}{2}(7 - 2\left(\frac{-23}{4}\right))$

$5 - \frac{1}{5}\left(\frac{-69}{4} - 4\right) = \frac{1}{2}(7 + \frac{46}{4})$

$5 - \frac{1}{5}\left(\frac{-69(1)-4(4)}{4(1)}\right) = \frac{1}{2}(\frac{7(4)+46(1)}{1(4)})$

$5 - \frac{1}{5}\left(\frac{-69-16}{4}\right) = \frac{1}{2}(\frac{28+46}{4})$

$5 - \frac{1}{5}\left(\frac{-85}{4}\right) = \frac{1}{2}(\frac{74}{4})$

$5 + \frac{85}{20} = \frac{74}{8}$

$\frac{5(20)+1(85)}{1(20)} = \frac{74}{8}$

$\frac{100+85}{20} = \frac{74}{8}$

$\frac{185}{20} = \frac{74}{8}$

$\frac{5.37}{5.4} = \frac{2.37}{2.4}$

$\frac{37}{4} = \frac{37}{4}$

1-2 Linear Inequalities

- ***Understanding Inequality and Interval Notation:***

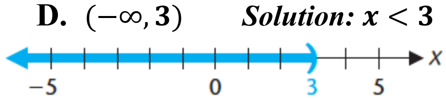
Ex: Rewrite each of the following** in inequality notation **and** graph **on a real number line:

A. $[-2, 3)$ ***Solution:*** $-2 \le x < 3$

B. $(-4, 2)$ ***Solution:*** $-4 < x < 2$

C. $[-2, \infty)$ ***Solution:*** $x \ge -2$

D. $(-\infty, 3)$ ***Solution:*** $x < 3$



E. $[-8, 7]$ ***Solution:*** $-8 \le x \le 7$

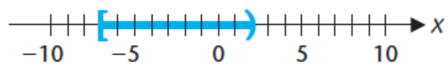
F. $[-6, \infty)$ ***Solution:*** $x \ge -6$

Ex. Rewrite each of the following *in interval notation* ***and*** *graph* ***on a real number line:***

A. $-2 < x \leq 6$ ***Solution:*** $(-2, 6]$ ***The graph:*** ……………………………………

B. $x \leq -2$ ***Solution:*** $(-\infty, -2]$ ***The graph:*** ……………………………………

Ex. Rewrite each of the following *in interval notation* ***and*** *in inequality notation:*

A. 

Solution: *in interval notation* $[-7, 2)$

in inequality notation $-7 \leq x < 2$

DEFINITION 2 Union and Intersection

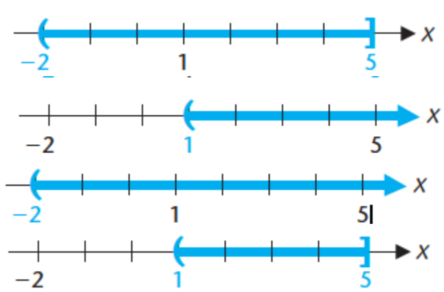
Union: $A \cup B = \{x \mid x \text{ is in } A \text{ or } x \text{ is in } B\}$

$\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

Intersection: $A \cap B = \{x \mid x \text{ is in } A \text{ and } x \text{ is in } B\}$

$\{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$

Ex. If $A = (-2, 5)$ ***and*** $B = (1, \infty)$, ***graph the sets*** $A \cup B$ ***and*** $A \cap B$ ***and write them in interval notation: Solution:***



$A = (-2, 5]$

$B = (1, \infty)$

$A \cup B = (-2, \infty)$

$A \cap B = (1, 5]$

- ***Solving Linear Inequalities:***

Ex. Solve and graph:

A. $7x - 8 < 4x + 7$

$7x - 4x < 7 + 8$

$3x < 15$

$\frac{3x}{3} < \frac{15}{3}$

$x < 5$, $x \in (-\infty, 5)$

The graph: ………………………………

B. $2(2x + 3) - 10 < 6(x - 2)$

$4x + 6 - 10 < 6x - 12$

$4x - 4 < 6x - 12$

$4x - 6x < -12 + 4$

$-2x < -8$

$\frac{-2x}{-2} > \frac{-8}{-2}$

$x > 4$, $x \in (4, \infty)$

The graph: 

C. $\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3}$

$$(12)\left(\frac{2x-3}{4}\right)+12(6) \geq 12(2)+(12)\left(\frac{4x}{3}\right)$$
$$3(2x-3)+72 \geq 24+4(4x)$$
$$6x-9+72 \geq 24+16x$$
$$6x+63 \geq 24+16x$$
$$6x-16x \geq 24-63$$
$$-10x \geq 39$$
$$\frac{-10x}{-10} \leq \frac{39}{-10}$$
$$x \leq \frac{39}{-10}, \quad x \in (-\infty, -3.9) \qquad \left(\frac{39}{-10} = -3.9\right)$$

D. $-3 \leq 4-7x < 1$

$$-3-4 \leq 4-4-7x < 18-4$$
$$-7 \leq -7x < 14$$
$$\frac{-7}{-7} \geq \frac{-7x}{-7} > \frac{14}{-7}$$
$$1 \geq x > -2 \qquad , x(-2, 1]$$

E. $-4 < 5t+6 \leq 21$

Solution: ………………………...

E $-6 \quad -4 < 5t+6 \leq 21 \quad -6$

$\frac{-10}{5} < \frac{5t}{5} \leq \frac{15}{5}$

$-2 < t \leq 3$

$(-2, 3]$

1-5 Quadratic Equations ($ax^2+bx+c=0, \ a \neq 0$)

➢ *Solving quadratic equations* ***by factoring:***

Ex: Solve each equation by factoring:

A. $(x-5)(x+3)=0$

Solution:

$x-5=0 \ \ Or \ \ x+3=0$

$x=+5 \ \ or \ \ x=-3$

Solution set $=\{-3, +5\}$

B. $6x^2-19x-7=0$

$(2x-7)(3x+1)=0$

$2x-7=0 \ \ Or \ \ 3x+1=0$

$2x=+7 \ \ or \ \ 3x=-1$

$\frac{2x}{2}=\frac{7}{2} \ \ Or \ \ \frac{3x}{3}=\frac{-1}{3}$

$x=\frac{7}{2} \ \ Or \ \ x=\frac{-1}{3}$

Solution set$=\{\frac{-1}{3}, \frac{7}{2}\}$

Check: $6x^2-19x-7=(2x-7)(3x+1)$
$=2x(3x+1)-7(3x+1)$
$=6x^2+2x-21x-7$
$=6x^2-19x-7$

C. $\boldsymbol{x^2-6x+5=-4}$

$x^2-6x+5+4=0$

$x^2-6x+9=0$

$(x-3)(x-3)=0$

$x-3=0\ \ Or\ \ x-3=0$

$x=+3\ \ Or\ \ x=+3$

Solution set= $\{3\}$

Check: $x^2-6x+9=(x-3)(x-3)$

$=x(x-3)-3(x-3)$

$=x^2-3x-3x+9$

$=x^2-6x+9$

D. $\boldsymbol{3w^2+13w=10}$ ***(Practice: the students can solve it by the same idea of C)***

E. $\boldsymbol{2x^2=3x}$

$2x^2-3x=0$

$x(2x-3)=0$

$x=0\ \ Or\ 2x-3=0$

$x=0\ \ 0r\ \ 2x=+3$

$x=0\ \ Or\ \frac{2x}{2}=\frac{3}{2}$

$x=0\ \ Or\ \ x=\frac{3}{2}$

Solution set= $\{0,\frac{3}{2}\}$

F. $\boldsymbol{2x^2=8x}$ ***(Practice: the students can solve it by the same idea of E)***

G. $\boldsymbol{-8=22t-6t^2}$

$6t^2-22t-8=0$

$(2t-8)(3t+1)=0$

$2t=+8\ \ Or\ \ 3t=-1$

$\frac{2t}{2}=\frac{8}{2}\ \ Or\ \ \frac{3t}{3}=\frac{-1}{3}$

$t=4\ \ Or\ \ t=\frac{-1}{3}$

Solution set= $\{\frac{-1}{3},4\}$

Check: $6t^2-22t-8=(2t-8)(3t+1)$

$=2t(3t+1)-8(3t+1)$

$=6t^2+2t-24t-8$

$=6t^2-22t-8$

- *Solving quadratic equations* ***by quadratic formula:***

If $\boldsymbol{ax^2+bx+c=0}$ **then** $\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

Ex. Solve each equation using the quadratic formula:

A. $\boldsymbol{x^2=3x+1}$

$x^2-3x-1=0$

$a=1,\ b=-3,\quad c=-1$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-1)}}{2(1)}$$

$$x=\frac{+3\pm\sqrt{9+4}}{2}$$

$$x=\frac{3\pm\sqrt{13}}{2}$$

Solution set= $\{\frac{3-\sqrt{13}}{2},\frac{3+\sqrt{13}}{2}\}$

B. $\boldsymbol{x^2 - 2x - 1 = 0}$

$a = 1, \quad b = -2, \qquad c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2 \pm \sqrt{4.2}}{2}$$

$$x = \frac{2 \pm \sqrt{4}\sqrt{2}}{2}$$

$$x = \frac{2 \pm 2\sqrt{2}}{2}$$

$$x = \frac{2(1 \pm \sqrt{2}\,)}{2}$$

$$x = 1 \pm \sqrt{2}$$

Solution set= $\{1 - \sqrt{2}\,, 1 + \sqrt{2}\}$

C. $\boldsymbol{2x + \frac{3}{2} = x^2}$

$$(2)(2x) + (2)\left(\frac{3}{2}\right) = (2)(x^2)$$

$4x + 3 = 2x^2$

$2x^2 - 4x - 3 = 0$

$a = 2, \quad b = -4, \ c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{+4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x = \frac{4 \pm \sqrt{4.10}}{4}$$

$$x = \frac{4 \pm \sqrt{4}\sqrt{10}}{4}$$

$$x = \frac{4 \pm 2\sqrt{10}}{4}$$

$$x = \frac{2(2 \pm \sqrt{10}\,)}{4}$$

$$x = \frac{2 \pm \sqrt{10}}{2}$$

Solution set= $\{\frac{2-\sqrt{10}}{2}, \frac{2+\sqrt{10}}{2}\}$

Ex. Solve each equation by any method:

A. $\mathbf{12x^2 + 7x = 10}$

$12x^2 + 7x - 10 = 0$

$(3x-2)(4x+5) = 0$

$3x - 2 = 0 \;\; Or \;\; 4x + 5 = 0$

$3x = +2 \;\; Or \;\; 4x = -5$

$\frac{3x}{3} = \frac{2}{3} \;\; Or \;\; \frac{4x}{4} = \frac{-5}{4}$

$x = \frac{2}{3} \;\; Or \;\; x = \frac{-5}{4}$

Solution set= $\{\frac{-5}{4}, \frac{2}{3}\}$

Check: $12x^2 + 7x - 10 = (3x-2)(4x+5)$
$= 3x(4x+5) - 2(4x+5)$
$= 12x^2 + 15x - 8x - 10$
$= 12x^2 + 7x - 10$

B. $\mathbf{(2y-3)^2 = 5}$

$2^2y^2 - 2(2y)(3) + 3^2 = 5$

$4y^2 - 12y + 9 = 5$

$4y^2 - 12y + 9 - 5 = 0$

$4y^2 - 12y + 4 = 0$

$a = 4, \quad b = -12, \;\; c = 4$

$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$y = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(4)}}{2(4)}$

$y = \frac{+12 \pm \sqrt{144-64}}{8}$

$y = \frac{12 \pm \sqrt{144-64}}{8}$

$y = \frac{12 \pm \sqrt{80}}{8}$

$y = \frac{12 \pm \sqrt{16.5}}{8}$

$y = \frac{12 \pm \sqrt{16}\sqrt{5}}{8}$

$y = \frac{12 \pm 4\sqrt{5}}{8}$

$y = \frac{4(3 \pm \sqrt{5})}{8}$

$y = \frac{3 \pm \sqrt{5}}{2}$

Solution set= $\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\}$

Chapter 3: Functions

3-1 Functions

- ***Definition of a function:***

Ex. Determine whether each set specifies a function. If it does, then state the domain and the range?

A. S= {(1,4), (2,3), (3,2), (4,3), (5,4)}

Solution: yes, it is a function, because all the ordered pairs in S have distinct first components.

$$Domain = \{1, 2, 3, 4, 5\}$$
$$Range = \{4,3,2\}$$

B. T= {(1,4), (2,3), (3,2), (2,4), (1,5)}

Solution: No, it is not a function, because there are ordered pairs in T with the same first component [for example, (1, 4) and (1, 5)].

C. {(10, -10), (5, -5), (0,0), (5,5), (10,10)} *(practice: the students can solve **C** by the same idea of **B**)*

Ex. Indicate whether each set defines a function. Find the domain and the range of each function?

A. {(-1,4), (0,3), (1,2), (2,1)}

Solution: yes, it is a function, because all the ordered pairs in this set have distinct first components.

$$Domain = \{-1, 0, 1, 2\}$$
$$Range = \{4, 3, 2, 1\}$$

B. {(2,4), (3,6), (4,8), (5,10)} *(practice: the students can solve **B** by the same idea of **A**)*

- ***Definition a function by equations:***

Ex:

$F(x) = x^2 - 4$

$y = x^2 - 4$

Evaluating Function:

Ex.1: **A.** Find $f(6)$, $f(a)$, and $f(6+a)$ for $\boldsymbol{f(x) = \frac{15}{x-3}}$

$$\boldsymbol{f(6)} = \frac{15}{\boldsymbol{6}-3} = \frac{15}{3} = 5$$

$$\boldsymbol{f(a)} = \frac{15}{\boldsymbol{a}-3}$$

$$\boldsymbol{f(6+a)} = \frac{15}{\boldsymbol{6+a}-3} = \frac{15}{a+3}$$

B. Find $g(7)$, $g(h)$, and $g(7+h)$ for $\boldsymbol{g(x) = 16 + 3x - x^2}$

$$\begin{aligned}\boldsymbol{g(7)} &= 16 + 3(\boldsymbol{7}) - (\boldsymbol{7})^2\\ &= 16 + 21 - 49\\ &= 37 - 49\\ &= -12\end{aligned}$$

$$\begin{aligned}\boldsymbol{g(h)} &= 16 + 3(\boldsymbol{h}) - \boldsymbol{h}^2\\ &= 16 + 3h - h^2\end{aligned}$$

$$\begin{aligned}\boldsymbol{g(7+h)} &= 16 + 3(\boldsymbol{7+h}) - (\boldsymbol{7+h})^2\\ &= 16 + 21 + 3h - (7^2 + 2(7)(h) + h^2)\\ &= 16 + 21 + 3h - (49 + 14h + h^2)\\ &= 16 + 21 + 3h - 49 - 14h - h^2\\ &= -12 - 11h - h^2\end{aligned}$$

C. Find $K(9)$, $4K(a)$, and $K(4a)$ $\quad for \quad \boldsymbol{K(x)} = \frac{2}{\sqrt{x}-2}$

$$\boldsymbol{K(9)} = \frac{2}{\sqrt{9}-2} = \frac{2}{3-2} = \frac{2}{1} = 2$$

$$\boldsymbol{4K(a)} = 4\left(\frac{2}{\sqrt{a}-2}\right) = \frac{4\cdot 2}{\sqrt{a}-2} = \frac{8}{\sqrt{a}-2}$$

$$\boldsymbol{K(4a)} = \frac{2}{\sqrt{4a}-2} = \frac{2}{\sqrt{4}\sqrt{a}-2} = \frac{2}{2\sqrt{a}-2} = \frac{2}{2(\sqrt{a}-1)} = \frac{1}{\sqrt{a}-1}$$

Ex.2: **A.** Let $\boldsymbol{f(x) = 3x - 5}$ $\quad find \quad f(3),\ f(h),\ f(3)+f(h),\ f(3+h)$

$\boldsymbol{f(3)} = 3(\boldsymbol{3}) - 5 = 9 - 5 = 4$

$\boldsymbol{f(h)} = 3\boldsymbol{h} - 5$

$\boldsymbol{f(3)} + \boldsymbol{f(h)} = 4 + (3h-5) = 4 + 3h - 5 = -1 + 3h$

$$\begin{aligned}\boldsymbol{f(3+h)} &= 3(\boldsymbol{3+h}) - 5\\ &= 9 + 3h - 5\\ &= 4 + 3h\end{aligned}$$

B. Let $f(w) = -w^2 + 2w$ $\quad find \quad f(4),\ f(-4),\ f(4+a),\ f(2-a)$

*(practice: the students can solve **B** by the same idea of **Ex.1 & Ex.2**)*

Finding the Domain of a Function:

Ex: Find the domain of the function defined by the equation: $y = \sqrt{x-3}$, assuming x is the independent variable.

Solution: for y to be real, $x - 3 \geq 0 \quad \Rightarrow x \geq +3$

∴ The domain of y is $x \geq 3 \quad or \quad x \in [3, \infty) \quad or \ \{x | x \geq 3\}$

Ex: Find the domain of each of the following function. Express the answer in both set notation and inequality notation?

A. $f(x) = \frac{15}{x-3}$ Solution: for $f(x)$ to be defined, $x - 3 \neq 0 \quad \Rightarrow x \neq 3$

∴ The domain of f is $x \neq 3 \quad or\ x \in (-\infty, 3) \cup (3, \infty) \quad or \ \{x | x \neq 3\}$

B. $g(x) = 16 + 3x + x^2$ Solution: The domain of g is $R \quad or \quad x \in (-\infty, \infty)$

C. $K(x) = \frac{2}{\sqrt{x}-2}$

Solution: 1. For $K(x)$ to be defined, $\sqrt{x} - 2 \neq 0 \quad \Rightarrow \sqrt{x} \neq 2 \quad \Rightarrow \left(\sqrt{x}\right)^2 \neq 2^2 \quad \Rightarrow x \neq 4$

2. For $K(x)$ to be real, $x \geq 0$

∴ The domain of K is $x \neq 4 \ and \ x \geq 0 \quad or\ x \in [0,4) \cup (4, \infty) \quad or \ \{x | x \neq 4 \ and \ x \geq 0\}$

Ex: Find the domain of each of the following function. Express the answer in both interval notation and inequality notation?

A. $f(x) = 4 - 9x + 3x^2$ ***Solution:*** The domain of $f(x)$ is $R \quad or \quad x \in (-\infty, \infty)$

B. $L(u) = \sqrt{3u^2 + 4}$ ***Solution:*** The domain of $L(u)$ is $R \quad or \quad u \in (-\infty, \infty)$

➢ *Even and Odd Functions:*

Definitions:

$If \ \boldsymbol{f(x)} = \boldsymbol{f(-x)} \quad for\ all\ x\ in\ the\ domain\ of\ f, then\ f\ is\ an\ \boldsymbol{even\ function}.$

$If \ -\boldsymbol{f(x)} = \boldsymbol{f(-x)} \quad for\ all\ x\ in\ the\ domain\ of\ f, then\ f\ is\ an\ \boldsymbol{odd\ function}$

Domain ⇒ المجال

كثيرات الحدود + الجذور الفردية

$f(x) = 3x^2 + 1$

$f(x) = \sqrt[5]{2x+1}$

$(-\infty, \infty+)$ أو $\mathbb{R}$

الجذور

$f(x) = \sqrt{x-5}$

مابداخل الجذر ≥ 0

$x - 5 \geq 0$

$x \geq 5$

$[5, \infty+)$

الكسور

$f(x) = \dfrac{3x-1}{x+4}$

$\mathbb{R} - \{$أصفار المقام$\}$

$x + 4 = 0$

$x = -4$

$\mathbb{R} - \{-4\}$

أو

$(-\infty, -4) \cup (-4, \infty+)$

Testing for Even and Odd Functions:

Ex.1: Determine whether the functions f, h , and g, are even, odd, or neither.

A. $f(x) = x^2 + 1$

Solution: $f(-x) = (-x)^2 + 1$

$= (-x)(-x) + 1$

$= x^2 + 1$

$= f(x)$

$\because f(-x) = f(x)$

$\therefore f$ is even.

B. $h(x) = x^5 + x$

Solution: $h(-x) = (-x)^5 + (-x)$

$= (-x)(-x)(-x)(-x)(-x) - x$

$= -x^5 - x$

$= -h(x)$ since $-h(x) = -(x^5 + x) = -x^5 - x$

$\because h(-x) = -h(x)$

$\therefore h$ is odd.

C. $g(x) = x^3 + 1$

Solution: $g(-x) = (-x)^3 + 1$

$= (-x)(-x)(-x) + 1$

$= -x^3 + 1$

$\neq g(x)$

$\because g(-x) \neq g(x)$

$\therefore g(x)$ is not even.

$-g(x) = -(x^3 + 1)$

$= -x^3 - 1$

$\neq g(-x)$

$\because -g(x) \neq g(-x)$

$\therefore g$ is not odd.

$\therefore g$ is neither even nor odd.

اختصار قاعدة الدوال الزوجية والفردية

زوجية even: $f(x) = x^2 + 1$, $f(x) = x^4 + 3x^2$ — كل الأسس فيها زوجية

odd فردية: $f(x) = x^7$, $f(x) = 3x^5 + 2x^3$ — الدالة فيها أسس فردية + لا يوجد فيها عدد بدون متغير

ليست زوجية ولا فردية: $f(x) = x^2 + x^5$ — الدالة فيها أس زوجي وأس فردي; $f(x) = x^5 - 1$ — الدالة فيها أس فردي وعدد

Ex.2: Indicate whether each of the following function is even, odd, or neither.

(practice: the students can solve **Ex.2** by the same idea of **Ex.1**)

A. $g(x) = x^3 + x$ = odd

Solution:

B. $m(x) = x^4 + 3x^2$ = Even

Solution:

C. $f(x) = x^5 + 1$ = neither even nor odd

Solution:

3-5 Operations on Functions; Composition

- ***Performing Operations on functions:***

Ex. *for the indicated functions f and g find the functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ and find their domains.*

A. $f(x) = 4x$; $g(x) = x + 1$

The domain of f is R or $x \in (-\infty, \infty)$

The domain of g is R or $x \in (-\infty, \infty)$

$f + g(x) = f(x) + g(x)$

$= 4x + x + 1$

$= 5x + 1$

The domain of $f + g$ is R or $x \in (-\infty, \infty)$

$f-g(x)=f(x)-g(x)$
$=4x-(x+1)$
$=4x-x-1$
$=3x-1$
The domain of $f-g$ *is* R *or* $x\in(-\infty,\infty)$

$fg(x)=f(x)g(x)$
$=4x(x+1)$
$=4x^2+4x$
The domain of fg *is* R *or* $x\in(-\infty,\infty)$

$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\frac{4x}{x+1}$
for f/g *to be defined,* $x+1\neq 0 \quad \Rightarrow x\neq -1$
The domain of f/g *is* $x\neq -1$ *or* $x\in(-\infty.-1)\cup(-1,\infty)$

B. $\boldsymbol{f(x)=2x^2}$; $\boldsymbol{g(x)=x^2+1}$ *(practice: the students can solve it by the same idea of A)*

Ex. *let* $f(x)=\sqrt{4-x}$ *and* $g(x)=\sqrt{3+x}$ *Find the functions:* $f+g,\ f-g,\ fg$ *and* $\frac{f}{g}$*, and find their domains.*

Solution: *For* $f(x)$ *to be real,* $4-x\geq 0 \quad \Rightarrow -x\geq -4 \quad \Rightarrow -\frac{x}{-1}\leq -\frac{4}{-1} \quad \Rightarrow x\leq 4$
The domain of f *is* $x\leq 4$ *or* $x\in(-\infty,4]$

For $g(x)$ *to be real,* $3+x\geq 0 \quad \Rightarrow x\geq -3$
The domain of g *is* $x\geq -3$ *or* $x\in[-3,\infty)$

$f+g(x)=f(x)+g(x)$
$=\sqrt{4-x}+\sqrt{3+x}$

$f-g(x)=f(x)-g(x)$
$=\sqrt{4-x}-\sqrt{3+x}$

$f.g(x)=f(x).g(x)$
$=\sqrt{4-x}\sqrt{3+x}$
$=\sqrt{(4-x)(3+x)}$
$=\sqrt{4(3+x)-x(3+x)}$
$=\sqrt{12+4x-3x-x^2}$
$=\sqrt{12+x-x^2}$
The domain of $f+g,\ f-g,$ *and* $f.g$ *is* $(-\infty,4]\cap[-3,\infty)=[-3,4]$

$\frac{f}{g}(x)=\frac{f(x)}{g(x)}$
$=\frac{\sqrt{4-x}}{\sqrt{3+x}}$
$=\sqrt{\frac{4-x}{3+x}}$
The domain of $\frac{f}{g}$ *is* $x\in(-3,4]$

Ex. let $\boldsymbol{f(x)=\frac{x}{x-1}}$***, and*** $\boldsymbol{g(x)=\frac{x-4}{x+3}}$ ***find the function*** $\boldsymbol{\frac{f}{g}}$ ***anf find its domain.***

Solution: *For* $f(x)$ *to be defined,* $x-1\neq 0 \quad \Rightarrow x\neq 1$
The domain of f *is* $x\neq 1$ *or* $x\in(-\infty,1)\cup(1,\infty)$

For $g(x)$ *to be defined,* $x+3\neq 0 \quad \Rightarrow x\neq -3$
The domain of g *is* $x\neq -3$ *or* $x\in(-\infty,-3)\cup(-3,\infty)$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x}{x-1} \div \frac{x-4}{x+3} = \frac{x}{x-1} \cdot \frac{x+3}{x-4} = \frac{x(x+3)}{(x-1)(x-4)}$$

$The\ domain\ of\ \frac{f}{g}\ \ is\ \ x \neq 1,\ \ x \neq 4\ \ and\ \ x \neq -3$

- ***Composition:***

Finding the Composition of Two Functions:

Ex: $Find\ \ (f \circ g)(x)\ \ for\ \ f(x) = x^2 - x\ \ \ and\ \ \ g(x) = 3 + 2x$

Solution:
$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3+2x) \\ &= (3+2x)^2 - (3+2x) \\ &= 3^2 + 2(3)(2x) + (2x)^2 - 3 - 2x \\ &= 9 + 12x + 4x^2 - 3 - 2x \\ &= 6 + 10x + 4x^2 \end{aligned}$$

Ex. $Find\ \ (f \circ g)(x)\ \ \ for\ \ f(x) = \sqrt{4-x^2}\ \ and\ \ g(x) = \sqrt{3-x},\ then\ find\ the\ domin\ of\ \ f \circ g.$

Solution: $The\ domain\ of\ \ f\ \ is\ \ \ -2 \leq x \leq 2\ \ \ or\ \ \ x \in [-2,2]$

$The\ domain\ of\ \ g\ \ is\ \ \ x \leq 3\ \ \ or\ \ \ x \in (-\infty, 3]$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(\sqrt{3-x}) \\ &= \sqrt{4 - \left(\sqrt{3-x}\right)^2} \\ &= \sqrt{4-(3-x)} \\ &= \sqrt{4-3+x} \\ &= \sqrt{1+x} \end{aligned}$$

$The\ domain\ of\ \ f \circ g\ \ is\ \ x \geq -1\ \ \ and\ \ \ x \leq 3\ \ \ or\ x \in [-1,3]$

Chapter10
10-3 Matrix Operations &10-4 Matrix Inverse method

10-3 Matrix Operations:

- ***Matrix Addition:***

Ex. *Add:*

A. $\underset{2\times 3}{\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix}} + \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix}} =$

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2+3 & -3+1 & 0+2 \\ 1+(-3) & 2+2 & -5+5 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

B. $\underset{2\times 3}{\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & -5 \end{bmatrix}} + \underset{3\times 2}{\begin{bmatrix} 0 & 2 \\ -3 & 5 \\ -1 & 4 \end{bmatrix}} =$

$Because\ the\ \textbf{size}\ of\ the\ first\ matrix\ is\ \mathbf{2\times 3}\ and\ the\ second\ is\ \mathbf{3\times 2}, this\ sum\ is\ not\ defined.$

- ***Matrix Subtraction:***

Ex. *Subtract:*

A. $\underset{2\times 2}{\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}} - \underset{2\times 2}{\begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix}} = \begin{bmatrix} 3-(-2) & -2-2 \\ 5-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & -4 \end{bmatrix}$

Matrix Equations:

Ex. $Find\ a, b, c, and\ d\ so\ that$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} a-2 & b-(-1) \\ c-(-5) & d-6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a-2 & b+1 \\ c+5 & d-6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

$a-2=4$	$b+1=3$	$c+5=-2$	$d-6=4$
$a=4+2$	$b=3-1$	$c=-2-5$	$d=4+6$
$a=6$	$b=2$	$c=-7$	$d=10$

- ***Multiplying a Matric by a Number:***

Ex. *Multiply:*

A. $-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(3) & -2(-1) & -2(0) \\ -2(-2) & -2(1) & -2(3) \\ -2(0) & -2(-1) & -2(-2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$

- ***Finding the Product of Two Matrices:***
  - ***Product of a Row Matrix and a Column Matrix:***

Ex. *Multiply:*

A. $\underset{1\times 3}{\begin{bmatrix} 2 & -3 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}} = [2(-5)+(-3)(2)+0(-2)] = [-10-6+0] = [-16]$

- ***Matrix Multiplication:***

Ex. $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Find each product that is defined:

A. $\boldsymbol{AB} = \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} = \begin{bmatrix} 2(1)+1(2) & 2(-1)+1(1) & 2(0)+1(2) & 2(1)+1(0) \\ 1(1)+0(2) & 1(-1)+0(1) & 1(0)+0(2) & 1(1)+0(0) \\ -1(1)+2(2) & -1(-1)+2(1) & -1(0)+2(2) & -1(1)+2(0) \end{bmatrix}$

$$= \begin{bmatrix} 2+2 & -2+1 & 0+2 & 2+0 \\ 1+0 & -1+0 & 0+0 & 1+0 \\ -1+4 & 1+2 & 0+4 & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 4 & -1 \end{bmatrix}$$

B. $\boldsymbol{BA} = \underset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} = product\ is\ not\ defined.$

C. $\boldsymbol{CD} = \underset{2\times 2}{\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} = \begin{bmatrix} 2(1)+6(3) & 2(2)+6(6) \\ -1(1)+(-3)(3) & -1(2)+(-3)(6) \end{bmatrix} = \begin{bmatrix} 2+18 & 4+36 \\ -1-9 & -2-18 \end{bmatrix} = \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$

D. $\boldsymbol{Dc} = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}} = \begin{bmatrix} 1(2)+2(-1) & 1(6)+2(-3) \\ 3(2)+6(-1) & 3(6)+6(-3) \end{bmatrix} = \begin{bmatrix} 2-2 & 6-6 \\ 6-6 & 18-18 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex. *Perform the indicated operations, if possible:*

A. $\underset{3\times 2}{\begin{bmatrix} \mathbf{4} & \mathbf{0} \\ \mathbf{-2} & \mathbf{3} \\ \mathbf{8} & \mathbf{1} \end{bmatrix}} + \underset{3\times 2}{\begin{bmatrix} \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{5} \\ \mathbf{4} & \mathbf{-6} \end{bmatrix}} = \begin{bmatrix} 4+(-1) & 0+2 \\ -2+0 & 3+5 \\ 8+4 & 1+(-6) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 8 \\ 12 & -5 \end{bmatrix}$

B. $\underset{2\times 3}{\begin{bmatrix} \mathbf{5} & \mathbf{-1} & \mathbf{0} \\ \mathbf{4} & \mathbf{6} & \mathbf{3} \end{bmatrix}} - \underset{2\times 3}{\begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{-6} \\ \mathbf{3} & \mathbf{5} & \mathbf{-5} \end{bmatrix}} = \begin{bmatrix} 5-2 & -1-4 & 0-(-6) \\ 4-3 & 6-5 & 3-(-5) \end{bmatrix} = \begin{bmatrix} 3 & -5 & 6 \\ 1 & 1 & 8 \end{bmatrix}$

C. $\mathbf{5}\begin{bmatrix} \mathbf{-7} & \mathbf{3} & \mathbf{0} & \mathbf{9} \\ \mathbf{4} & \mathbf{-5} & \mathbf{6} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} 5(-7) & 5(3) & 5(0) & 5(9) \\ 5(4) & 5(-5) & 5(6) & 5(2) \end{bmatrix} = \begin{bmatrix} -35 & 15 & 0 & 45 \\ 20 & -25 & 30 & 10 \end{bmatrix}$

D. $\begin{bmatrix} \mathbf{-2} & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{3} \\ \mathbf{-8} \end{bmatrix} = [-2(3)+4(-8)] = [-6-32] = [-38]$

10-4 Matrix Inverse method

- ***The Identity Matrix for multiplication:*** $IM = MI = M$

Ex. $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 1(5)+0(2) & 1(3)+0(1) \\ 0(5)+1(2) & 0(3)+1(1) \end{bmatrix} = \begin{bmatrix} 5+0 & 3+0 \\ 0+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

➢ ***Finding the Inverse of a Square Matrix:***

$If\ A\ is\ a\ 2\times 2\ matrix\ given\ by\quad A=\begin{bmatrix} a & b\\ c & d\end{bmatrix},\qquad then\ A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$

Ex.1 $Given\ A, find\ A^{-1}, if\ it\ exists, Check\ each\ inverse\ by\ showing\ AA^{-1}=I.$

A. $\boldsymbol{A}=\begin{bmatrix} \mathbf{1} & \mathbf{9}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$$

$$A^{-1}=\frac{1}{1(1)-9(0)}\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}=\frac{1}{1-0}\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}=\frac{1}{1}\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}=1\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}=\begin{bmatrix} 1(1) & 1(-9)\\ 1(0) & 1(1)\end{bmatrix}=\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}$$

$Check:\ AA^{-1}=I$

$$\begin{bmatrix} 1 & 9\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & -9\\ 0 & 1\end{bmatrix}=\begin{bmatrix} 1(1)+9(0) & 1(-9)+9(1)\\ 0(1)+1(0) & 0(-9)+1(1)\end{bmatrix}=\begin{bmatrix} 1+0 & -9+9\\ 0+0 & 0+1\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}=I$$

B. $\boldsymbol{A}=\begin{bmatrix} \mathbf{-5} & \mathbf{7}\\ \mathbf{2} & \mathbf{-3}\end{bmatrix}$

Solution:

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$$

$$A^{-1}=\frac{1}{-5(-3)-(7)(2)}\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}=\frac{1}{15-14}\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}=\frac{1}{1}\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}=1\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}=\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}$$

Check: $AA^{-1}=I$

$$\begin{bmatrix} -5 & 7\\ 2 & -3\end{bmatrix}\begin{bmatrix} -3 & -7\\ -2 & -5\end{bmatrix}=\begin{bmatrix} -5(-3)+7(-2) & -5(-7)+7(-5)\\ 2(-3)+(-3)(-2) & 2(-7)+(-3)(-5)\end{bmatrix}=\begin{bmatrix} 15-14 & 35-35\\ -6+6 & -14+15\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}=I$$

C. $\boldsymbol{A}=\begin{bmatrix} \mathbf{4} & \mathbf{-1}\\ \mathbf{-6} & \mathbf{2}\end{bmatrix}$ *(practice: the students can solve* **C** *by the same idea of* **A &B)**

D. $\boldsymbol{A}=\begin{bmatrix} \mathbf{10} & \mathbf{-2}\\ \mathbf{-5} & \mathbf{1}\end{bmatrix}$

Solution:

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$$

$$A^{-1}=\frac{1}{10(1)-(-2)(-5)}\begin{bmatrix} 1 & -(-2)\\ -(-5) & 10\end{bmatrix}$$

$$A^{-1}=\frac{1}{10-10}\begin{bmatrix} 1 & 2\\ 5 & 10\end{bmatrix}=\frac{1}{0}\begin{bmatrix} 1 & 2\\ 5 & 10\end{bmatrix}$$

$\because ad-bc=10(1)-(-2)(-5)=10-10=0$

$\therefore\ A^{-1}\ does\ not\ exist.$

Statistics

Frequency Distributions and Graphs:

Organizing Data:

Ex.1 *Twenty-five army inductees were given a blood test to determine their blood type. The data set is*

A	B	B	AB	O
O	O	B	AB	B
B	B	O	A	O
A	O	O	O	AB
AB	A	O	B	A

*Construct **a frequency distribution** for the data.*

Solution:

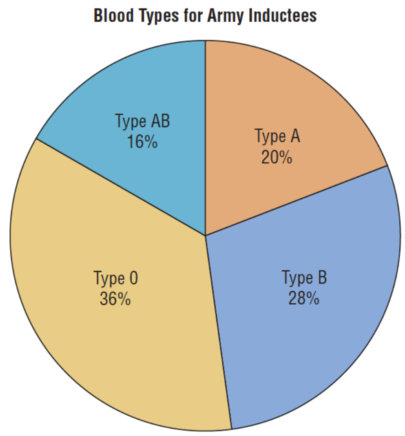
Class	Tally	Frequency(f)	Percent= $\frac{f}{n} \times 100\%$
A	~~||||~~	5	5/25=0.20$\times$ 100 = 20%
B	~~||||~~ //	7	7/25=0.28$\times$ 100 = 28%
O	~~||||~~ ////	9	9/25=0.36$\times$ 100 = 36%
AB	////	4	4/25=0.16$\times$ 100 = 16%
Total		25	100

Ex.2 *Construct **a pie graph** showing the blood types of the army inductees described in the previous example. The frequency distribution is repeated here.*

Class	Frequency	Percent
A	5	20
B	7	28
O	9	36
AB	4	16
Total	25	100

Solution:

Class	Frequency	Percent	Degree = $\frac{f}{n} \times 360°$
A	5	20	$\frac{5}{25} \times 360° = 72°$
B	7	28	$\frac{7}{25} \times 360° = 100.8°$
O	9	36	$\frac{9}{25} \times 360° = 129.6°$
AB	4	16	$\frac{4}{25} \times 360° = 57.6°$
Total	25	100	360°



Exer. *Trust in Internet Information A survey was taken on how much trust people place in the information they read on the Internet. Construct **a categorical frequency distribution** for the data. A = trust in everything they read, M = trust in most of what they read, H = trust in about one-half of what they read, S = trust in a small portion of what they read. (Based on information from the UCLA Internet Report.)*

M	M	M	A	H	M	S	M	H	M
S	M	M	M	M	A	M	M	A	M
M	M	H	M	M	M	H	M	H	M
A	M	M	M	H	M	M	M	M	M

*(practice: the students can solve **this exercise** by the same idea of **Ex.1**)*

Grouped Frequency Distributions:

Ex.1 *These data represent the record high temperatures in degrees Fahrenheit (%F) for each of the 50 states. Construct **a grouped frequency distribution** and **a cumulative frequency distribution** for the data using 7 classes.*

112	100	127	120	134	118	105	110	109	112
110	118	117	116	118	122	114	114	105	109
107	112	114	115	118	117	118	122	106	110
116	108	110	121	113	120	119	111	104	111
120	113	120	117	105	110	118	112	114	114

Solution:

The highest value :H =134 and the lowest value: L = 100.

The range: R = highest value − lowest value = H – L

R = 134 − 100 = 34

$$Width = \frac{Range}{number\ of\ classes} = \frac{34}{7} = 4.9$$

Class Limits	Class boundaries	Tally	Frequency
100–104	99.5–104.5		2
105–109	104.5–109.5		8
110–114	109.5–114.5		18
115–119	114.5–119.5		13
120–124	119.5–124.5		7
125–129	124.5–129.5		1
130–134	129.5–134.5		1

$$n = \sum f = 50$$

	Cumulative frequency
Less than 99.5	0
Less than 104.5	2
Less than 109.5	10
Less than 114.5	28
Less than 119.5	41
Less than 124.5	48
Less than 129.5	49
Less than 134.5	50

Exer. *State Gasoline Tax The state gas tax in cents per gallon for 25 states is given below. Construct* ***a grouped frequency distribution*** *and* ***a cumulative frequency distribution*** *with 5 classes.*

7.5	16	23.5	17	22
21.5	19	20	27.1	20
22	20.7	17	28	20
23	18.5	25.3	24	31
14.5	25.9	18	30	31.5

Source: *The World Almanac and Book of Facts.*

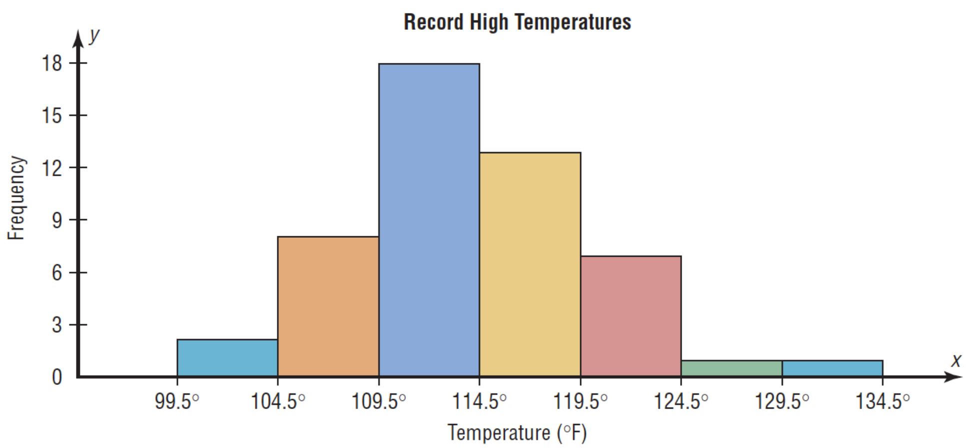
(practice: the students can solve ***this exercise*** *by the same idea of* ***Ex.1****)*

Histograms, Frequency polygons, Ogives:

Ex. *Construct* ***a histogram****,* ***a frequency polygon****, and* ***an ogive*** *to represent the data shown for the record high temperatures for each of the 50 states.*

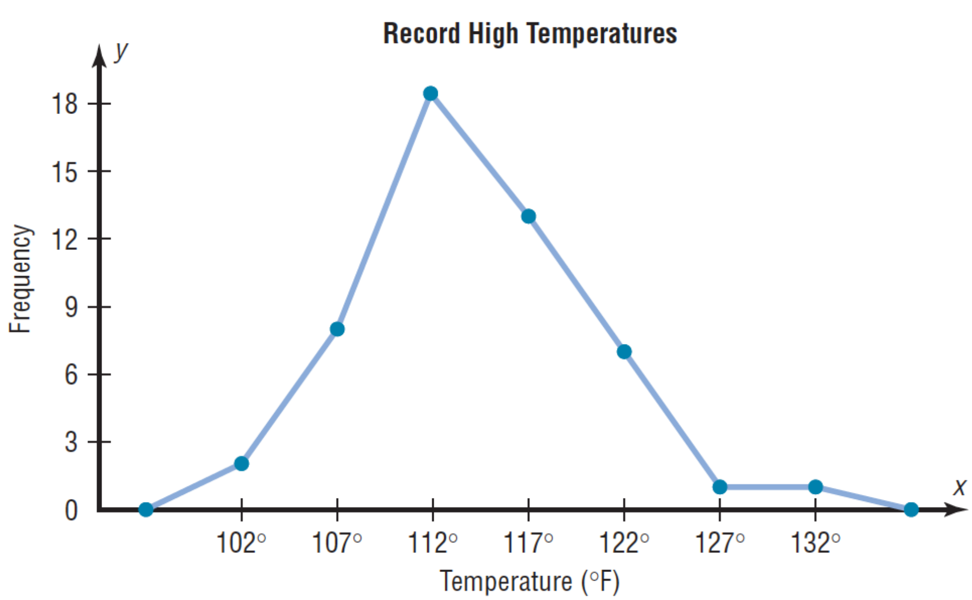
Class boundaries	Frequency
99.5–104.5	2
104.5–109.5	8
109.5–114.5	18
114.5–119.5	13
119.5–124.5	7
124.5–129.5	1
129.5–134.5	1

Solution:
The Histogram:



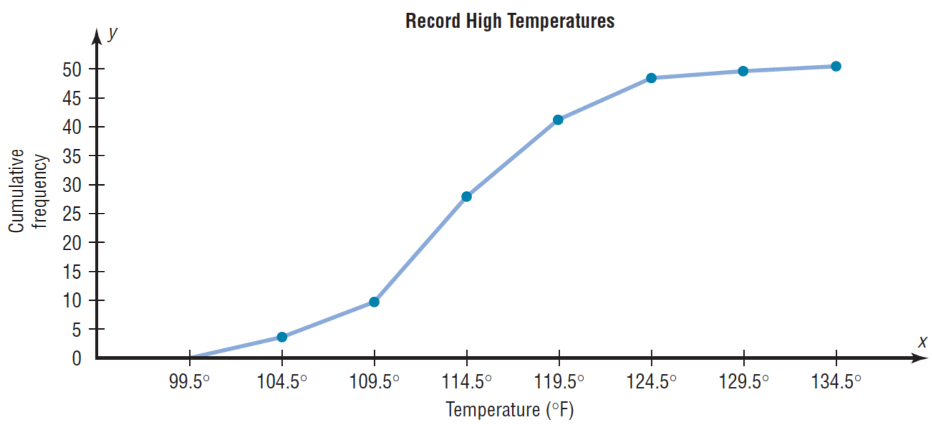
The Frequency Polygon:

Class boundaries	Frequency	Midpoints ($X_m = \frac{Lower\ boundary + Upper\ boundary}{2}$ ***or*** $X_m = \frac{Lower\ limit + upper\ limit}{2}$)
99.5–104.5	2	$\frac{99.5+104.5}{2} = 102$
104.5–109.5	8	$\frac{104.5+109.5}{2} = 107$
109.5–114.5	18	112
114.5–119.5	13	117
119.5–124.5	7	122
124.5–129.5	1	127
129.5–134.5	1	132



The Ogive:

	Cumulative frequency
Less than 99.5	0
Less than 104.5	2
Less than 109.5	10
Less than 114.5	28
Less than 119.5	41
Less than 124.5	48
Less than 129.5	49
Less than 134.5	50

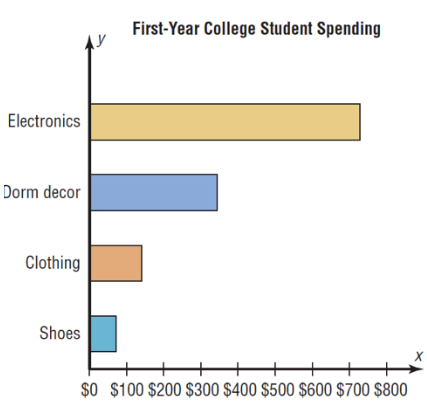


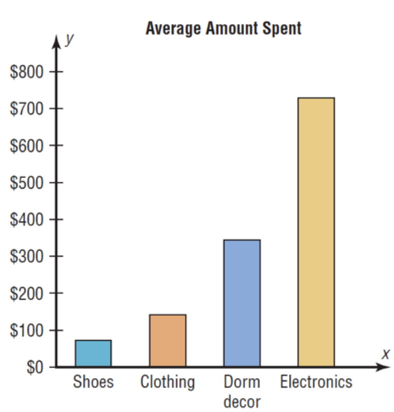
Bar Graphs:

Ex. *The table shows the average money spent by first-year college students. Draw a horizontal and vertical bar graph for the data.*

Electronics	$728
Dorm décor	344
Clothing	141
Shoes	72

Solution:





Data Description:

The Mean:

Ex.1 *The data represent the number of days off per year for a sample of individuals selected from nine different countries. Find the mean.* *20, 26, 40, 36, 23, 42, 35, 24, 30*

Solution: $\overline{x} = \frac{x_1+x_2+x_3+\cdots+x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$\overline{x} = \frac{x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9}{9} = \frac{20+26+40+36+23+42+35+24+30}{9} = \frac{276}{9} = 30.7\ days$$

Ex.2 *The data shown represent the number of boat registrations for six counties in southwestern Pennsylvania. Find the mean.* *3782 6367 9002 4208 6843 11,008*

Solution: *(practice: the students can solve **Ex.2** by the same idea of **Ex.1**)*

The Median:

Ex.1 *The number of rooms in the seven hotels in downtown Pittsburgh is 713, 300, 618, 595, 311, 401, and 292. Find the median.*

Solution: ***1.*** *292, 300, 311, 401, 595, 618, 713.*

→ → → ↑ ← ← ←

2. *Median=MD=401*

Ex.2 *The number of tornadoes that have occurred in the United States over an 8-year period follows. Find the median.* *684, 764, 656, 702, 856, 1133, 1132, 1303*

Solution: ***1.*** *656, 684, 702, 764, 856, 1132, 1133, 1303.*

→ → → → ↑ ← ← ← ←

2. *Median=MD=*$\frac{764+856}{2} = \frac{1620}{2} = 810$

The Mode:

Ex.1 *Find the mode of the signing bonuses of eight NFL players for a specific year. The bonuses in millions of dollars are* *18.0, 14.0, 34.5, 10, 11.3, 10, 12.4, 10*

Solution: It is helpful to arrange the data in order although it is not necessary.

10, 10, 10, 11.3, 12.4, 14.0, 18.0, 34.5

The Mode= 10 *(Since $10 million occurred 3 times)*

Ex.2 *Find the mode for the number of coal employees per county for 10 selected counties in southwestern Pennsylvania.* *110, 731, 1031, 84, 20, 118, 1162, 1977, 103, 752*

*Solution: Since each value occurs only once, there **is no mode**.*

Ex.3 *The data show the number of licensed nuclear reactors in the United States for a recent 15-year period. Find the mode.*

104	104	104	104	104
107	109	109	109	110
109	111	112	111	109

Solution: The modes = 104 and 109.

The Midrange:

Ex.1 *In the last two winter seasons, the city of Brownsville, Minnesota, reported these numbers of water-line breaks per month. Find the midrange. 2, 3, 6, 8, 4, 1*

Solution: $Midrange = MR = \frac{lowest\ value(L) + highest\ value(H)}{2}$

$$= \frac{1+8}{2} = \frac{9}{2} = 4.5$$

Ex.2 *Find the midrange of the following data: 18.0, 14.0, 34.5, 10, 11.3, 10, 12.4, 10*

Solution: *(practice: the students can solve **Ex.2** by the same idea of **Ex.1**)*

Exer. *The average undergraduate grade point average (GPA) for the 25 top-ranked medical schools is listed below.*

3.80	3.77	3.70	3.74	3.70
3.86	3.76	3.68	3.67	3.57
3.83	3.70	3.80	3.74	3.67
3.78	3.74	3.73	3.65	3.66
3.75	3.64	3.78	3.73	3.64

Find (a) the mean, (b) the median, (c) the mode, and (d) the midrange.

Solution:

The mean= $\bar{x} = \frac{x_1+x_2+x_3+\cdots+x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$= \frac{x_1+x_2+x_3+\cdots+x_{25}}{25} = \frac{3.80+3.77+3.70+3.74+3.70+3.86+\cdots+3.64}{25} = \frac{93.09}{25} = \mathbf{3.72}$$

The median=MD=

1. *3.57*, 3.64, 3.64, 3.65, 3.66, 3.67, 3.67, 3.68, *3.70*, *3.70*, *3.70*, 3.73, **3.73**, *3.74*, *3.74*, *3.74*, 3.75, 3.76, 3.77, 3.78, 3.78, 3.80, 3.80, 3.83, *3.86*.

2. The Median=MD= 3.73

The mode= 3.70 *and* 3.74

.

The midrange= $MR = \frac{lowest\ value(L)+highest\ value(H)}{2} = \frac{\mathbf{3.57+3.86}}{\mathbf{2}} = \frac{\mathbf{7.43}}{\mathbf{2}} = \mathbf{3.72}$

References:

Barnett, R., & Ziegler, M., Precalculus, 7th Ed., McGraw-Hill, New York, 2011.
https://drive.google.com/file/d/1MRjNLXaVsNK4ibkaACpRgsseXUP7Q5FR/view

Bluman, Allan G., Elementary Statistics: A step approach, 7th Ed., McGraw-Hill, New York, 2009.
https://drive.google.com/file/d/1jjCwuZJaf1JXyGj2svR73eJ1nEHjqK29/view

Larson, R. & Hostetler, R., Precalculus, 7th Ed., Houghton Mifflin, Boston, 2007.